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# The Principle of Temperate Action and a New Type of Uncertainty Relation

JERZY RAYSKI

*Institute of Physics, Jagellonian University, Cracow, Poland*

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*Abstract*

A new physical principle, called Principle of Temperate Action, is introduced to secure the convergence of the theory of quantised fields. In order to satisfy this principle, as well as the correspondence requirements, a dimensional constant **a** must be incorporated into the Lagrangian, which becomes bounded or increasing sufficiently slowly for its arguments tending to infinity. In some cases of temperate actions there appear constraints introducing non-local features and/or uncertainties of the order of magnitude of the constant **a**, in spite of the fact that the Lagrangians are local.

## 1. *The Principle of Temperate Action*

In order to formulate a more satisfactory theory of particles and their interactions it is not necessary to abandon or modify any of the first principles constituting the foundations of the traditional theory of quantised fields, but only to supplement them by a fundamental assumption of a restrictive character. Thus, we preserve the following principles.

The field equations are to be derived from a variational principle with a local Lagrangian density $\mathscr{L}(x)$ dependent upon the arguments $x_\mu$ via the field quantities and their first-order derivatives. The Lagrangian must be of a form that allows for a canonical formulation and quantisation whereby the Hamiltonian must be real and positive definite. Moreover, in situations where gravitation can be neglected, the Lagrangian has to satisfy the requirements of Special Relativity, i.e. has to be a scalar with respect to the Poincaré group.

On the other hand, in order to avoid the peculiarities of the traditional theories we have to look for quite unusual types of Lagrangians and Hamiltonians. It is easy to show that if the Hamiltonian density $\mathscr{H}$ is positive definite and may be majorised by another Hamiltonian density $\mathscr{H}_0$ quadratic in the field quantities and their derivatives, i.e. if there exist two numbers $M$, $N$ so that

$$0 \leqslant \mathscr{H} \leqslant M\mathscr{H}_0 + N \tag{1.1}$$

for all values of their arguments $\phi$, $\phi_{,\mu}$, then such a theory is free of the usual convergence difficulties. In other words, we get a theory free of the usual divergences if the Hamiltonian does not increase faster than quadratically in terms of its arguments tending to infinity. In quantum theory the inequalities (1.1) have to be understood as holding for expectation values in an arbitrary state.

*Proof*: Integrating over the three-dimensional space† the inequalities (1.1) are seen to hold true also for the global Hamiltonians $H$ and $H_0$. Taking a sequence of orthonormal eigenstates of $H_0$ (representing a set of free fields) to its eigenvalues $E_n^{(0)}$ we get, from the well-known theorem of Eckhart,

$$E_n \leqslant \langle E_n^{(0)} | H | E_n^{(0)} \rangle < ME_n^{(0)} + N \tag{1.2}$$

This result, combined with the requirement of positive definiteness, shows that to each finite eigenvalue of $H_0$ there exists at least one finite eigenvalue of $H$ satisfying the inequalities

$$0 \leqslant E_n \leqslant ME_n^{(0)} + N \tag{1.2a}$$

Thus, $H$ is an acceptable operator and, consequently, the development of the system in the course of time, described by the unitary operator $\exp(iHt)$ is free of any inconsistencies.

The Hamiltonians $\mathscr{H}$ satisfying (1.1) may be called temperate and the restrictions (1.1) may be raised to the rank of a fundamental principle, to be called the Principle of Temperate Action (PTA).

The traditional field theories with Lagrangians and Hamiltonians splitting into two parts, so that one (describing free fields) is quadratic, and the other (describing the interaction) is a polynomial of degree three or higher (i.e. involves products of at least three field quantities), do not satisfy our principle. In order to satisfy both PTA and the requirements of correspondence with the traditional field theories (in the limit of weak fields) it is necessary to incorporate into the Lagrangian a fundamental constant $a$ with dimension $cm^4$ in units $c = \hbar = 1$.

Examples of field theories consistent with the PTA were investigated first by Fradkin (1963) and Efimov (1963a, b, 1965) and, more recently, by Hoegh-Krohn (1967) and the present author (Rayski, 1969a, b). One of the simplest examples is provided by a real scalar field $\phi$ interacting with itself via a non-linear interaction Lagrangian

$$\mathscr{L}' = -g\phi^4(1 + a\phi^4)^{-\alpha} \tag{1.3}$$

where $\alpha \geqslant \frac{1}{2}$. The quantity $M$ appearing in (1.1) may be chosen in this case to be

$$M = 1 + |g|\, m^{-2} a^{-1/2} \tag{1.4}$$

In the case of repulsive forces ($g > 0$) the energy is always positive definite, while in the case of attractive forces ($g < 0$) it is so if $|g|$ is sufficiently small

† We consider the field in a finite box of periodicity in order to deal with discrete eigenvalues of energy.

or the bare mass $m$ is sufficiently large (the renormalised mass may be quite small). A sufficient condition for the positive definiteness of $H$ is, in this case,

$$|g| < m^2 a^{1/2} \tag{1.5}$$

In spite of the fact that such a theory is free of the usual convergence difficulties, it did not evoke much interest—for the following reasons:

(i) It seemed to be extremely difficult to handle any practical problems in the case of so complicated non-linear interactions.
(ii) It was argued that, still, there are essential difficulties with the construction of a non-trivial $S$-operator.
(iii) The way of tempering the interaction by modifying $\mathscr{L}'$ in a way similar to (1.3) does not seem promising because it cannot be applied to electrodynamics. Indeed, any modification of $\mathscr{L}'$ would destroy the gauge invariance.

In connection with point (i) it could be argued that the Lagrangians of the type (1.3) or similar are discouraging because the only way to handle them in practice, is to develop them into a power series and to curtail the series after a finite number of terms. But any finite polynomial means an untemperate interaction violating the condition (1.1), so that we run again into the usual convergence difficulties. However, as pointed out previously (Rayski, 1969a, b), a neglection of higher powers of the expansion of (1.3) makes sense only in the case of weak fields, and has to be combined with a restriction upon the class of state vectors: the state vectors to be taken into account in a given approximation have to represent sufficiently weak fields. The remaining state vectors (even those only involved in virtual states) have to be disregarded completely. By increasing the range of state vectors to be used in a computation we also have to increase suitably the number of terms of the power series expansion of $\phi$. Thus, there exists a simple method of computations starting with a weak field approximation.

The troubles with the problem of constructing a non-trivial $S$-operator are not connected with any particular form of quantum field theory but with the fact that the mere concept of an $S$-operator is meaningless. As I have shown previously (Rayski, 1969c), the usual definitions of the $S$-operator (e.g. as that transforming the ingoing free fields into outgoing free fields) are incorrect because the corresponding matrix elements mean probability amplitudes in this representation only where $s$ is diagonal. Besides, the basis vectors of this representation are not normalisable to unity and do not belong to the Hilbert space. The only correct operator describing the evolution of the system in the course of time is $\exp[-iH(t_2 - t_1)]$, which obviously does not admit a transition to the limits $t_{1/2} \to \mp\infty$. The amplitudes for transitions between different states characterised by sharp values of the invariants $s$, $t$, $u$ acquire only a meaning by careful limiting procedure, starting with non-overlapping wave packets and going over to the limit $\Delta s \to 0$ at the very end of the calculation.

In order to reconcile electrodynamics with our postulate (1.1) we have to modify the Lagrangian as a whole, or, at least, to modify it without destroying its gauge-invarant parts: $\mathscr{L}_{em}$ denoting the Lagrangian of the electromagnetic field, and $\mathscr{L}_{ch}$ denoting the Lagrangian of the charged field including the interaction. Thus,

$$\mathscr{L} = F(\mathscr{L}_{em}, \mathscr{L}_{ch}) \tag{1.6}$$

where $F$ is a function to be chosen so as to satisfy (1.1) as well as the requirement of correspondence with the traditional electrodynamics. The function $F$ may depend also upon some other gauge-invariant terms, e.g. upon $(\bar{\psi}\gamma_\mu \psi)^2$. In particular, we may try the following generalisation

$$\mathscr{L} = \frac{1}{a} \operatorname{arc} tga\Lambda \tag{1.6a}$$

where $\Lambda$ is the traditional Lagrangian

$$\Lambda = \mathscr{L}_{em} + \mathscr{L}_{ch} \tag{1.6b}$$

However, in this case there appear several problems as to the possibility of a canonical formulation and quantisation of the theory of fields described by such unusual Lagrangians. We shall illustrate these problems with the aid of a few examples.

## 2. *Examples of Unusual Lagrangians*

The traditional field theories always exhibited the following features (often believed to be necessary conditions for quantisation). The range of variability of the field quantities and their first-order derivatives was assumed to be unlimited. The Lagrangians were chosen to be real functions defined in the whole domain of variability of their arguments $\phi$ and $\phi_{,\mu}$. The choice of the Lagrangians was restricted by the following requirement: it should be possible to resolve the formulae

$$\pi = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \tag{2.1}$$

uniquely in the form

$$\dot{\phi} = \dot{\phi}(\pi, \phi, \phi_{,k}) \tag{2.2}$$

enabling a canonical formulation.† The momenta $\pi$ were always real if $\phi$ were real and the range of their variability was unlimited. The Hamiltonian density could be defined as

$$\mathscr{H} = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \dot{\phi} - \mathscr{L} \tag{2.3}$$

† In special cases one encounters a situation that the right-hand sides of the relations (2.1) are independent of some of the $\dot{\phi}$'s. These degrees of freedom (field quantities) give rise to some equations of constraints and are called variables of constraints in contradistinction to the remaining genuine dynamical variables.

and expressed uniquely as a function of $\pi$, $\phi$ and their space-like derivatives. It represented a real function for arbitrary values of its arguments $\pi$, $\phi$, $\phi_{,k}$.

With the aid of a few examples we shall show that it is possible to abandon, or weaken, the above assumptions without destroying the possibility of obtaining a mathematically consistent and physically acceptable quantum theory.

In the non-relativistic mechanics it was always assumed that a free particle is described by a Lagrangian quadratic in the velocity

$$L = \tfrac{1}{2}\dot{q}^2 \tag{2.4}$$

(only one degree of freedom $q = m^{1/2}x$ is taken into consideration). However, the Lagrangian does not need to be as simple as that, even in the case of a free particle, if a fundamental constant $a$, dimensionally independent of $c$ and $h$, plays a fundamental role in Nature, but it may involve the constant $a$ in one way or another. Let us consider as an example

$$L = \frac{1}{a}[1 - \sqrt{(1 - a\dot{q}^2)}] \tag{2.5}$$

going over into (2.4) in the limit $a \to 0$. If $a$ is positive and we demand (2.5) to be real, then there appears a restriction upon the velocity

$$\dot{q}^2 \leqslant \frac{1}{a} \tag{2.6}$$

This restriction means one-sided constraints. Obviously, the general solution of the Lagrange equation following from (2.5) represents a motion with a constant velocity

$$q(t) = At + B \tag{2.7}$$

where $A = \dot{q}$ is subjected to the restriction (2.6).

This theory may be put into a canonical form by postulating the usual relation

$$p = \frac{\partial L}{\partial \dot{q}} \tag{2.8}$$

and demanding $p$ to be real. The solution of (2.8) with respect to $\dot{q}$ is

$$\dot{q} = \frac{p}{\sqrt{(1 + ap^2)}} \tag{2.8a}$$

and satisfies automatically the restriction (2.6). Defining the energy in the usual way

$$H = \frac{\partial L}{\partial \dot{q}}\dot{q} - L \tag{2.9}$$

and introducing (2.8a) we get

$$H(p,q) = \frac{1}{a}[\sqrt{(1 + ap^2)} - 1] \tag{2.9a}$$

which is positive definite but otherwise unlimited. The Hamilton equations are equivalent to the Lagrange equation and to the relation (2.8) or (2.8a). Thus, the canonical formulation is not only possible but also automatically takes account of the constraints (2.6). Nothing prevents us from quantising such a theory.

The above example possesses a close field-theoretical analogue if we assume the following Lagrangian

$$\tilde{\mathscr{L}} = \frac{2}{a}[1 - \sqrt{(1 - a\mathscr{L}_0)}] \tag{2.10}$$

where $\mathscr{L}_0$ is a conventional Lagrangian, e.g.

$$\mathscr{L}_0 = \tfrac{1}{2}[\dot{\phi}^2 - (\operatorname{grad}\phi)^2 - m^2\phi^2] \tag{2.10a}$$

In this case (2.1) becomes

$$\pi = \frac{\partial\tilde{\mathscr{L}}}{\partial\dot{\phi}} = \frac{\dot{\phi}}{\sqrt{(1 - a\mathscr{L})}} \tag{2.11}$$

with the solution

$$\dot{\phi} = \pi\left(1 + \frac{a}{2}(\operatorname{grad}\phi)^2 + \frac{a}{2}m^2\phi^2\right)^{1/2}\left(1 + \frac{a}{2}\pi^2\right)^{-1/2} \tag{2.12}$$

satisfying automatically the restriction

$$\dot{\phi}^2 < (\operatorname{grad}\phi)^2 + m^2\phi^2 + \frac{2}{a} \tag{2.13}$$

When trying to introduce additionally an interaction, e.g.

$$\mathscr{L}' = q\phi^4 \tag{2.14}$$

we have to make the replacement

$$\mathscr{L}_0 \to \mathscr{L}_0 + \mathscr{L}' \tag{2.15}$$

but not to add $\mathscr{L}'$ to $\tilde{\mathscr{L}}$. Inasmuch as the interaction term (2.14) appears under the square root, the Lagrangian increases quadratically for $\phi$ tending to infinity, so that the action is temperate.

Going over to field quantisation we face the problem of how to define functions of the operators $\phi$, $\phi_{,k}$ and $\pi$, which are not polynomials but infinite series. It was pointed out that such operator-valued distributions are not well defined. However, these objections are intimately connected with the use of the Fock representation, and only show that this representation is illegitimate. There exist other representations, inequivalent to Fock's representation, for which a broader class of functions of the field operators may be defined.

Let us consider another example of a non-conventional Lagrangian

$$\tilde{\mathscr{L}} = \mathscr{L} + V(\phi^2) \tag{2.16}$$

where $\mathscr{L}$ is a conventional Lagrangian whereas

$$V(\phi^2)=\begin{cases}\infty & \text{for} \quad \phi^2>\frac{1}{a}\\ 0 & \text{for} \quad \phi^2\leqslant\frac{1}{a}\end{cases} \tag{2.17}$$

This theory is closely analogous to the case of a particle in a box in ordinary quantum mechanics, where the restriction $|q|<L$ was satisfied by restricting the wave functions by a condition

$$\langle q'|\Psi\rangle=0 \quad \text{for } |q'|\geqslant L \tag{2.18}$$

In close analogy a restriction upon the space of states

$$\langle q'|\rangle=0 \quad \text{for} \quad \varphi^2>\frac{1}{a} \tag{2.19}$$

may be introduced in the $\phi$-representation where the states are functionals of the field quantity $\phi$, whereas the momentum $\pi$ is represented by a functional derivative

$$\pi(\vec{x})=\frac{1}{i}\frac{\delta}{\delta\phi(\vec{x})} \tag{2.20}$$

in the Schrödinger picture. In this case the $\phi$-representation is certainly privileged, whereas the Fock representation acquires (approximately) a sense in the limit of weak fields.

Inasmuch as the space of states $\langle\phi'|\rangle$ is restricted by (2.19), it is impossible to construct a state characterised by a value of $\pi$ fixed with an arbitrary accuracy. The momentum canonically conjugate to the field quantity is no more observable in the usual sense but there appears a new type of a (single) uncertainty relation

$$\Delta\pi\geqslant\sqrt{a} \tag{2.21}$$

A similar restriction appears if one considers a Lagrangian of the type

$$\mathscr{L}=\mathscr{L}_0+\frac{g}{a}\sqrt{(1-a\phi^4)} \tag{2.22}$$

## 3. *Conclusion*

Let us conclude with the following remarks. Special Relativity may be characterised by the fact that it takes proper account of the existence of a dimensional constant $c$ playing a manifestly restrictive role ($v\leqslant c$). Also, Quantum Theory consists in a consequent incorporation of a dimensional constant $h$ into the framework of physical theories, whereby the role of $h$ is also restrictive (namely, Heisenberg's uncertainty relations). Now, there appears a possibility of taking proper account of a third fundamental constant $a$, dimensionally independent of $c$ and $h$ and playing also a manifestly restrictive role by tempering the action and introducing non-local features into the theory, in spite of the fact that the Lagrangians are perfectly local.

Inasmuch as every dynamical quantity becomes dimensionless if expressed in natural units $c = h = a = 1$ (or acquires a natural unit of its own), the incorporation of the third fundamental constant, so as to modify essentially the Lagrangian, may be regarded as completing this development of physics whose main steps in the past were a transition from pre-relativistic to relativistic and from classical to quantum theories.

Of course, we are still very far from the final goal, because we do not know yet the exact form of the World-Hamiltonian but only some general framework for self-consistent theories. Nevertheless, the fact that, now, we are able to construct many examples of quantised interacting fields, free of any inconsistencies, may be regarded as constituting some progress, in comparison with the situation of yesterday, where only one case was known with certainty to be free of internal contradictions: the case of free fields.

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